Madras College Maths Department
Higher Maths
R&C 1.4 Integration

Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Indefinite Integrals

In **integration**, our aim is to "undo" the process of differentiation. Later we will see that integration is a useful tool for evaluating areas and solving a special type of equation.

We have already seen how to differentiate polynomials, so we will now look at how to undo this process. The basic technique is:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1, \ c \text{ is the constant of integration.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add the constant of integration (c).

EXAMPLES

1. Find $\int x^2 \, dx$.

2. Find $\int x^{-3} \, dx$.

3. Find $\int x^{\frac{5}{4}} \, dx$.

- We use the symbol $\int$ for integration.
- The $\int$ must be used with "dx" in the examples above, to indicate that we are integrating with respect to x.
- The constant of integration is included to represent any constant term in the original expression, since this would have been zeroed by differentiation.
- Integrals with a constant of integration are called **indefinite integrals**.

Checking the answer

Since integration and differentiation are reverse processes, if we differentiate our answer we should get back to what we started with.

For example, if we differentiate our answer to Example 1 above, we do get back to the expression we started with.

$$\begin{tikzcd} \frac{1}{3}x^3 + c \arrow[r, bend left, "\text{differentiate}"] & x^2 \arrow[l, bend left, "\text{integrate}"] \end{tikzcd}$$

Integrating terms with coefficients

The above technique can be extended to:

$$\int ax^n \, dx = a\int x^n \, dx = \frac{ax^{n+1}}{n+1} + c \qquad n \neq -1, \ a \text{ is a constant.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add on c.

EXAMPLES

4. Find $\int 6x^3 \, dx$.

5. Find $\int 4x^{-\frac{3}{2}} \, dx$.

Note

It can be easy to confuse integration and differentiation, so remember:

$$\int x \, dx = \tfrac{1}{2}x^2 + c \qquad\qquad \int k \, dx = kx + c.$$

Other variables

Just as with differentiation, we can integrate with respect to any variable.

EXAMPLES

6. Find $\int 2p^{-5}\,dp$.

Note
dp tells us to integrate with respect to p.

7. Find $\int p\,dx$.

Note
Since we are integrating with respect to x, we treat p as a constant.

Integrating several terms

The following rule is used to integrate an expression with several terms:

$$\int \left(f(x) + g(x) \right) dx = \int f(x)\,dx + \int g(x)\,dx.$$

Stated simply: integrate each term separately.

EXAMPLES

8. Find $\int \left(3x^2 - 2x^{\frac{1}{2}} \right) dx$.

9. Find $\int \left(4x^{-\frac{5}{8}} + 3x + 7 \right) dx$.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Preparing to Integrate

As with differentiation, it is important that before integrating, all brackets are multiplied out, and there are no fractions with an x term in the denominator (bottom line), for example:

$$\frac{1}{x^3}=x^{-3} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}} \qquad \frac{1}{4x^5}=\frac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2}=\frac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Find $\int \frac{dx}{x^2}$ for $x \neq 0$.

2. Find $\int \frac{dx}{\sqrt{x}}$ for $x > 0$.

3. Find $\int \frac{7}{2p^2}\,dp$ where $p \neq 0$.

4. Find $\int \frac{3x^5 - 5x}{4} \, dx$.

5 $\int \frac{7x-3x^2\sqrt{x}}{\sqrt{x}} \, dx$

A Special Integral

The method for integrating an expression of the form $(ax+b)^n$ is:

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \qquad \text{where } a \neq 0 \text{ and } n \neq -1.$$

Stated simply: raise the power (n) by one, divide by the new power and also divide by the derivative of the bracket $(a(n+1))$, add c.

EXAMPLES

1. Find $\int (x+4)^7 \, dx$.

2. Find $\int (2x+3)^2 \, dx$.

3. Find $\int \frac{1}{\sqrt[3]{5x+9}} \, dx$ where $x \neq -\frac{9}{5}$.

4 $\int \frac{7}{2(1-3x)^5}dx$

Warning

Make sure you don't confuse differentiation and integration – this could lose you a lot of marks in the exam.

Remember the following rules for differentiation and integrating expressions of the form $(ax+b)^n$:

$$\frac{d}{dx}\left[(ax+b)^n\right] = an(ax+b)^{n-1},$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

These rules will *not* be given in the exam.

Using Differentiation to Integrate

Recall that integration is the process of undoing differentiation. So if we differentiate $f(x)$ to get $g(x)$ then we know that $\int g(x)\,dx = f(x) + c$.

EXAMPLES

5. (a) Differentiate $y = \dfrac{5}{(3x-1)^4}$ with respect to x.

   (b) Hence, or otherwise, find $\displaystyle\int \frac{1}{(3x-1)^5}\,dx$.

6. (a) Differentiate $y = \frac{1}{\left(x^3 - 1\right)^5}$ with respect to x.

(b) Hence, find $\int \frac{x^2}{\left(x^3 - 1\right)^6}\, dx$.

Integrating $\sin(ax+b)$ and $\cos(ax+b)$

Since we know the derivatives of $\sin(ax+b)$ and $\cos(ax+b)$, it follows that the integrals are:

$$\int \cos(ax+b)\ dx = \tfrac{1}{a}\sin(ax+b) + c\,,$$

$$\int \sin(ax+b)\ dx = -\tfrac{1}{a}\cos(ax+b) + c\,.$$

These are given in the exam.

EXAMPLES

1. Find $\int \sin(4x+1)\ dx$.

2. Find $\int \cos\left(\tfrac{3}{2}x + \tfrac{\pi}{5}\right)\ dx$.

Find $\int 2\cos\left(\tfrac{1}{2}x - 3\right)\ dx$.

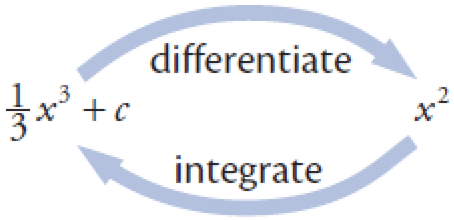
Find $\int 5\cos(2x)+\sin\left(x-\sqrt{3}\right)\,dx$.

(a) Differentiate $\dfrac{1}{\cos x}$ with respect to x.

(b) Hence find $\displaystyle\int \frac{\tan x}{\cos x}\,dx$.

Differential Equations

A **differential equation** is an equation involving derivatives, e.g. $\frac{dy}{dx} = x^2$.

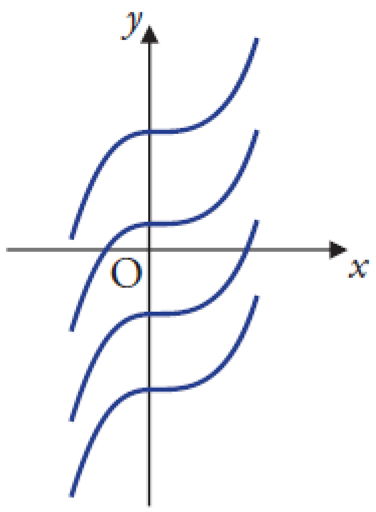


A solution of a differential equation is an expression for the original function; in this case $y = \frac{1}{3}x^3 + c$ is a solution.

In general, we obtain solutions using integration:

$$y = \int \frac{dy}{dx}\, dx \qquad \text{or} \qquad f(x) = \int f'(x)\, dx.$$

This will result in a **general solution** since we can choose the value of c.



The general solution corresponds to a "family" of curves, each with a different value for c.

The graph to the left illustrates some of the curves $y = \frac{1}{3}x^3 + c$ with different values of c.

If we have additional information about the function (such as a point its graph passes through), we can find the value of c and obtain a **particular solution**.

EXAMPLES

1. The graph of $y = f(x)$ passes through the point $(3, -4)$.

   If $\frac{dy}{dx} = x^2 - 5$, express y in terms of x.

2. The function f, defined on a suitable domain, is such that

   $f'(x) = x^2 + \frac{1}{x^2} + \frac{2}{3}$.

   Given that $f(1) = 4$, find a formula for $f(x)$ in terms of x.

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Definite Integrals

RC

If $F(x)$ is an integral of $f(x)$, then we define:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

where a and b are called the **limits** of the integral.

Stated simply:

- Work out the integral as normal, leaving out the constant of integration.
- Evaluate the integral for $x = b$ (the upper limit value).
- Evaluate the integral for $x = a$ (the lower limit value).
- Subtract the lower limit value from the upper limit value.

Since there is no constant of integration and we are calculating a numerical value, this is called a **definite integral**.

EXAMPLES

1. Find $\int_1^3 5x^2\,dx$.

2. Find $\int_0^2 \left(x^3 + 3x^2\right) dx$.

3. Find $\displaystyle\int_{-1}^{4} \frac{4}{x^3}\,dx$.

4 $\displaystyle\int_{1}^{4} \frac{2x\sqrt{x} + 3\sqrt{x}}{x^3}$

Definite Intervals Involving Trigonometric Functions

1 Find the value of $\int_0^1 \cos(2x-5)\,dx$.

2 $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\cos(x-\frac{\pi}{6})\,dx$

3 Find the values of a, $\pi \le a \le 3\pi/2$ for which:

$$\int_{\pi}^{a} 2\cos 2x - \sin x \, dx = 1.$$

Practice Unit Assessments

Practice test 1

1 Find $\int \left(4x^{\frac{1}{3}} + \frac{1}{x^3}\right) dx,\ x > 0.$

[4]

2 $h'(x) = (x+5)^{-4}$ find $h(x),\quad x \neq -5$.

[2]

3 Find $\int 4\cos\theta\, d\theta.$

[1]

4 Find $\int_{-3}^{2} \left(x^2 - 8x + 16\right) dx$.

[3]

Practice test 2

1 Find $\int (2x^{\frac{1}{3}} + \frac{1}{x^4})\ dx,\ x \neq 0.$

[4]

2 $h'(x) = (x-3)^{-2}$, find $h(x)$, $x \neq 3$.

[2]

3 Find $\int 2\sin\theta\, d\theta$. .

[1]

4 Find $\int_{-3}^{1} (x+2)^3\, dx.$

[3]

Homework

1

Find $\int 8\cos(4x+1)\,dx$.

2

SQA Higher Maths 2016 Non Calc Q5

2

(a) Find $\int (3\cos 2x+1)\,dx$. **2**

(b) Show that $3\cos 2x + 1 = 4\cos^2 x - 2\sin^2 x$. **2**

(c) Hence, or otherwise, find $\int (\sin^2 x - 2\cos^2 x)\,dx$. **2**

SQA Higher Maths 2015 Calc Q7

3

Given that $\int_4^t (3x+4)^{-\frac{1}{2}}\,dx = 2$, find the value of t. **5**

SQA Higher Maths 2014 Calc Q5

4

Find the value of $\int_0^2 \sin(4x+1)\,dx$. **4**

SQA Higher Maths 2007 Calc Q7

5

For a function f, defined on a suitable domain, it is known that:

- $f'(x) = \dfrac{2x+1}{\sqrt{x}}$
- $f(9) = 40$

Express $f(x)$ in terms of x. **4**

SQA Higher Maths 2016 Calc Q9

Practice Unit Assessment Solutions (1)

(1) $\int 4x^{\frac{1}{3}} + \frac{1}{x^3} \, dx$

$= \int 4x^{\frac{1}{3}} + x^{-3} \, dx$

$= \frac{4x^{4/3}}{4/3} + \frac{x^{-2}}{-2} + c$

$= \frac{3}{4} \cdot 4x^{4/3} - \frac{x^{-2}}{2} + c$

$= 3x^{4/3} - \frac{1}{2x^2} + c$

(2) $h(x) = \int h'(x) \, dx$

$= \int (x+5)^{-4} \, dx$

$= \frac{(x+5)^{-3}}{1 \times -3} + c$

$= -\frac{(x+5)^{-3}}{3} + c$

$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(3) $\int 4\cos\theta \, d\theta$

$= 4\sin\theta + c$

④ $\int_{-3}^{2} x^2 - 8x + 16 \, dx$

$= \left[\frac{x^3}{3} - \frac{8x^2}{2} + 16x\right]_{-3}^{2}$

$= \left[\frac{x^3}{3} - 4x^2 + 16x\right]_{-3}^{2}$

$= \left(\frac{(2)^3}{3} - 4(2)^2 + 16(2)\right) - \left(\frac{(-3)^3}{3} - 4(-3)^2 + 16(-3)\right)$

$= \left(18\frac{2}{3}\right) \cancel{+} - (-93)$

$= 111\frac{2}{3} \quad \left(\text{or } \frac{335}{3}\right)$

Practice Unit Assessment Solutions (2)

① $\int 2x^{\frac{1}{3}} + x^{-4} \, dx$

$= \frac{2x^{4/3}}{4/3} + \frac{x^{-3}}{-3} + c$

$= \frac{3}{4} . 2x^{4/3} - \frac{x^{-3}}{3} + c$

$= \frac{3x^{4/3}}{2} - \frac{1}{3x^3} + c$

(2) $h(x) = \int h'(x)\,dx$

$= \int (x-3)^{-2}\,dx$

$= \dfrac{(x-3)^{-1}}{1 \times -1} + c$

$\int (ax+b)^n = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$

$= \underline{\underline{-(x-3)^{-1} + c}}$

(3) $\int 2\sin\theta\,d\theta$

$= \underline{\underline{-2\cos\theta + c}}$

(4) $\int_{-3}^{1} (x+2)^3\,dx$

$= \left[\dfrac{(x+2)^4}{1 \times 4}\right]_{-3}^{1}$

$= \dfrac{(1+2)^4}{4} - \dfrac{(-3+2)^4}{4}$

$= \dfrac{81}{4} - \dfrac{1}{4}$

$= \underline{\underline{20}}$